4. NONLINEAR DYNAMICAL SYSTEMS

4.1 Application of Lie Groups and TSC to Investigate Chaos

Chaos was first observed in the Duffing's equation by Ueda. Recently, Vaidya and Anderson (JASA, May 1991) have developed the Trans-Spectral Coherence (TSC) analysis technique and applied it to the chaotic signals generated by this equation. The technique consists of looking at coherence across the frequency spectrum. It has been found to be particularly useful in analysing signals generated by non-linear mechanisms.

Using TSC technique, considerable amount of order was uncovered in the chaotic signal of the Duffing's equation. Although, it has been well known for a long time that chaos is quite distinct from purely random signal, the actual values of coherence were quite surprising. For example, nearly unity trans-coherence was found between the main driving frequency and all it's odd harmonics up to 17. The even harmonics had relatively low coherence. Both these results were quite puzzling for a long time.

A theoretical study has now been carried out which not only explained these results but led to considerably greater insight about the nature of the chaos, itself. The first step in the method consisted of embedding the equation in the four-dimensional space to arrive at an autonomous form. This, and a related three-dimensional form have the interesting property that the equations can now be cast in the framework of the Lie groups. These groups possess certain "symmetries", which means that once one solution is known several other solutions can be readily predicted. For the Duffing's equation, two types of symmetries are used to find two subgroups of solutions. Even during chaos, the trajectories remain confined to these subgroups.

These results have been used to explain all the TSC results, mentioned above. They have also led to suggestions of the alternative TSC results which have also been verified. The investigation has also led to an understanding of bifurcations and its role in creating a chaotic signal, and the nature of the chaos itself. *(P.G. Vaidya, P. Prasad*, *IISc, Bangalore).*

4.2 Electrocardiogram (ECG) Data Analysis

In recent years some controversy has arisen about normal ECG signals. It has been claimed that the signals are chaotic, if not random. ECG data, obtained from a normal human being is analysed. The results show that this is true. The analysis has also been able to educe considerable order in the midst of chaos.

It is found that when the intervals between the peaks of a typical ECG signal (Fig. 4.1a) are plotted as the $(M+1)^{st}$ interval vs. M^{th} interval (Fig. 4.1b), the peak distribution shows a chaotic behaviour. Further, a conditional sampling, using the knowledge of these locations show considerable order (Fig. 4.1c). The new TSC is much larger than the TSC by the straight forward analysis.

Further analysis has been carried out which shows that even the random looking intervals can be predicted from the data in earlier records. An iterative formulation has been remarkably successful at predicting future records. The main issue now is, how do the coefficients of the recursive functions change as arrythmias and myocardial infractions approach? *(P. Vaidya, N. Pradhan*, *NIMHANS, Bangalore).*

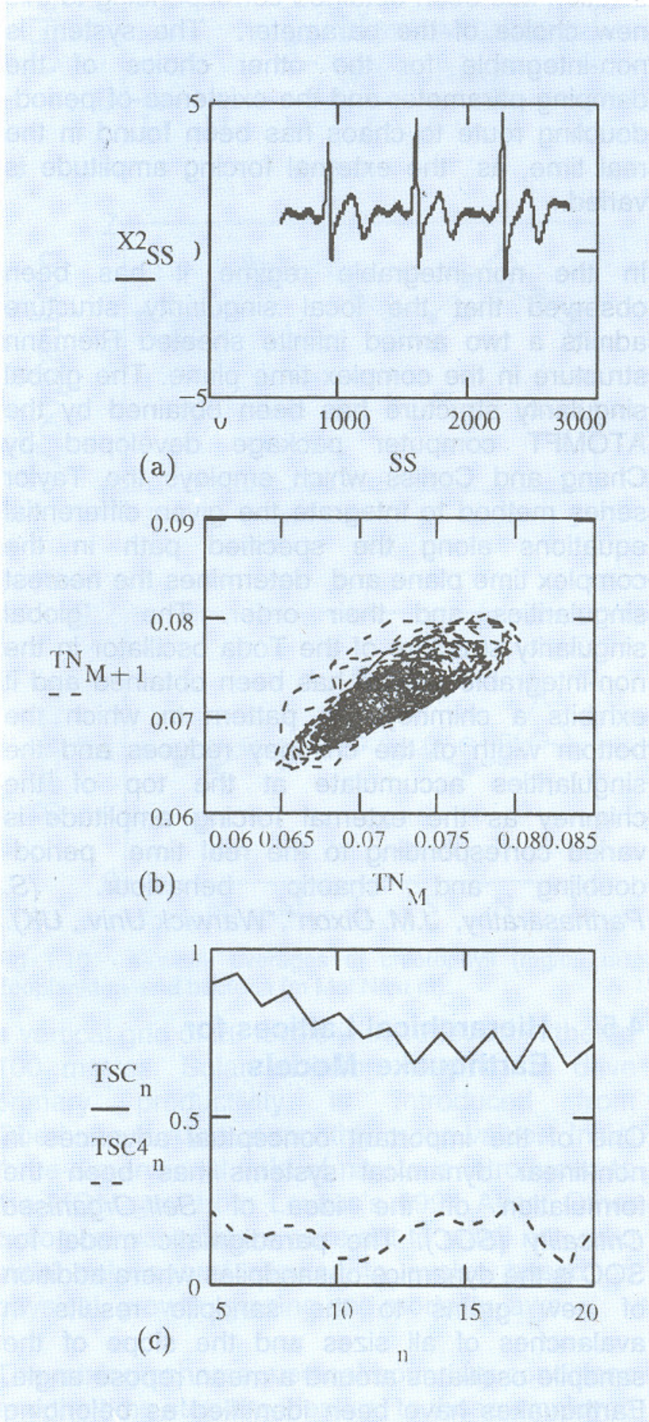

Fig. 4.1: (a) Typical ECG data, (b) (M+1)st interval Vs. Mth interval, (c) TSC on sampled ECG (full line) and older TSC (dashed line)

4.3 Electroencephalogram (EEG) and Seismic Data Analysis

TSC analysis of EEG data (Fig. 4.2a) and seismic data (Fig. 4.2b) have been carried out. The coherence values are seen to be low. This is explained by, for example, the FFT spectrum of two consecutive records of the EEG data, shown in Fig. 4.2c. It can be seen that the spectrum changes quite rapidly in between these records. These changes cause leakage in spectral amplitudes and phases. Such leakage is quite harmful to the TSC. Similar results are obtained for seismic data.

We therefore need a method to correct for leakage, without harming the phase information, and a method of frequency renormalization. A method to achieve this in the limited case of spectra which are dominated by a single frequency has been completed. *(P. G. Vaidya, N. Pradhan*, V.K. Gaur, *NIMHANS, Bangalore).*

4.4 Damped Driven Toda Oscillator: Singularity Structure Analysis and Chaotic Dynamics

Most natural processes are governed by non-linear (partial/ordinary) differential equations. In order to understand the evolution of a given process it is necessary to know the solutions of its associated non-linear equations. There are no standard techniques to obtain solutions of any given set of non-linear differential equations as in the case of linear equations. So if it is possible to identify the parameter values of a given system analytically by some method for which the system is integrable then it will be very useful. The Painleve Singularity structure analysis (or Ablowitz, Ramani and Segur algorithm) is a systematic algorithmic procedure to identify whether the given set of non-linear differential equations governing a dynamical system is of Painleve type (and hence is likely to integrable) or not. The idea is to expand the solution of a n^{th} order non-linear differential equation as a Laurent's series in the neighbourhood of a movable singularity and find (assumed to one of the arbitrary integration constants) whether

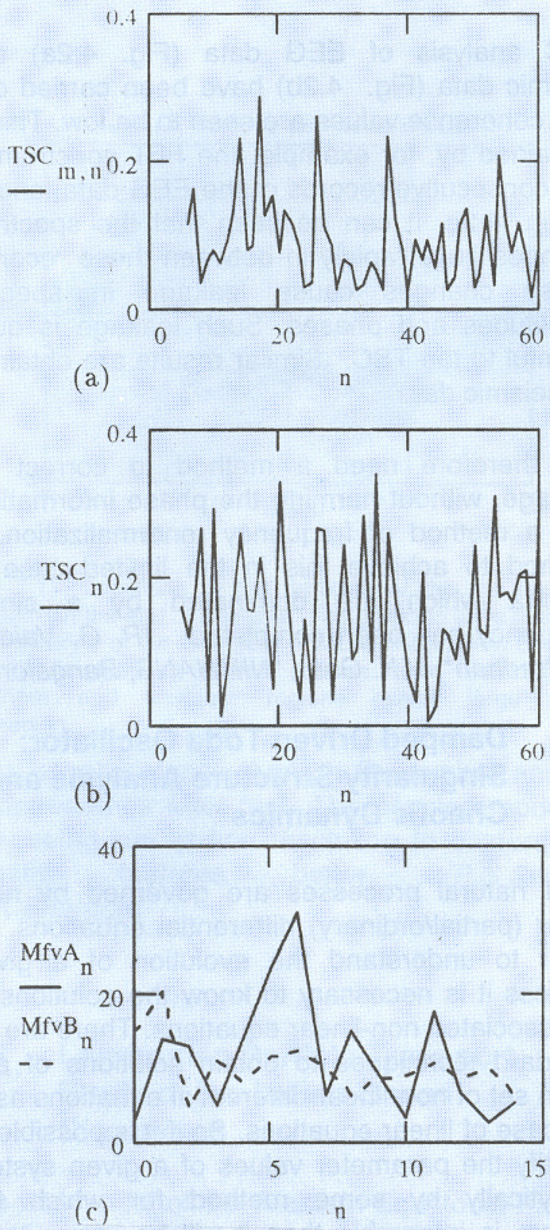

Fig. 4.2: (a) TSC of EEG data, (b) TSC of data from Killari earthquake, (c) Spectra of two consecutive records of the EEG.

there exists sufficient number (say, *n-1*) of arbitrary coefficients or not. This method has been applied to the equation of motion of the damped driven Toda oscillator and it has been found that there exists a specific choice of the damping parameter for which the free Toda oscillator possesses Painleve property and hence it is likely to be integrable. The exact solution has been obtained corresponding to this new choice of the parameter. The system is non-integrable for the other choice of the damping parameter and the existence of period-doubling route to chaos has been found in the real time, as the external forcing amplitude is varied.

In the non-integrable regime it has been observed that the local singularity structure admits a two armed infinite sheeted Riemann structure in the complex time plane. The global singularity structure has been obtained by the ATOMFT computer package developed by Chang and Corliss which employs the Taylor series method to integrate the given differential equations along the specified path in the complex time plane and determines the nearest singularities and their order. The global singularity structure of the Toda oscillator in the non-integrable regime has been obtained and it exhibits a chimney like pattern in which the bottom width of the chimney reduces and the singularities accumulate at the top of the chimney as the external forcing amplitude is varied corresponding to the real time period-doubling and chaotic behaviour. *(S. Parthasarathy, J.M. Dixon*,*Warwick Univ., UK).*

4.5 Hierarchical Lattices for Earthquake Models

One of the important conceptual advances in non-linear dynamical systems has been the formulation of the idea of *Self-Organised Criticality (SOC)*. The paradigmatic model for SOC is the dynamics of sandpiles where addition of new grains to the sandpile results in avalanches of all sizes and the slope of the sandpile oscillates around a mean repose angle. Earthquakes have been identified as belonging to this class of non-linear phenomena with the dynamic fluctuations being the small stress drops in the earthquake events around a mean tectonic stress and the distribution of sizes of events following a power law with respect to the frequency of events in an analogous fashion to the avalanches in sandpiles. The preliminary results from the study of non-linear dynamical system versions of earthquake models show that the statistical distributions in the dynamics approximate the expected distributions with just a homogeneous distribution of blocks modelling the faults. The question then arises as to what the real heterogeneities in the lithosphere contribute to the dynamics. Lithosphere has a hierarchical, fractal structure extending across many decades. Models involving heterogeneous distribution of blocks are being studied with a view to developing models that can be used specifically with reference to particular faults by incorporating the particular heterogeneous structure of those faults into the model. Varieties of hierarchical lattices are being tried out to see the differences in the event patterns that they give rise to. It is anticipated that these studies will also help in the understanding of the associated inverse problem of connecting a particular event pattern with a particular fault structure and thereby leading to better prediction/hazard quantification schemes. *(T. R. Krishna Mohan).*

4.6 Modelling of Biochemical Oscillations

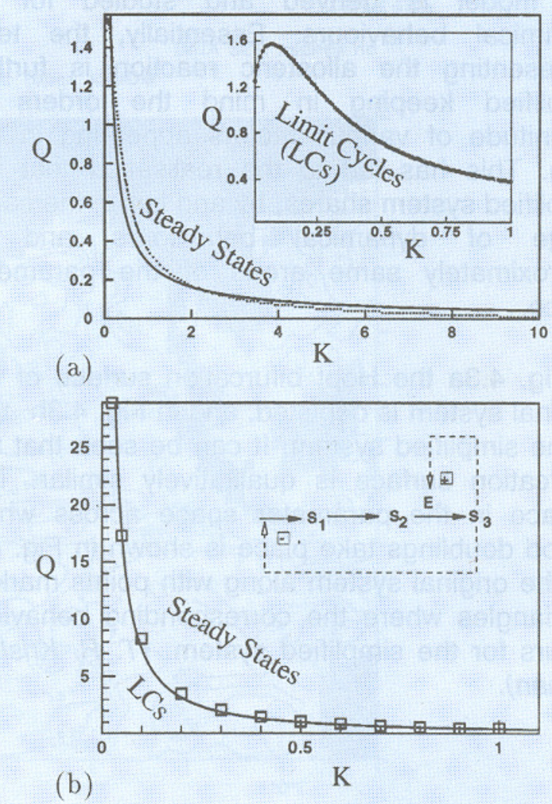

Fig. 4.3: Hopf bifurcation surfaces of (a) the original, and (b) the simplified versions of the biochemical pathway model. The solid lines in both the figures are power law fits to the data points. Inset to (a) shows an enlarged portion, close to the origin, of the surface. Inset to (b) shows a schematic of the metabolic pathway comprising of a coupled system of positive and negative feedback loops.

Biorhythms manifest itself at all levels from the molecular to macroscopic with periods ranging from fractions of a second to years. However, oscillations do not always have a simple periodic nature. For example, in neural systems it can take the appearance of complex burstings followed by regular periods of quiescence. There are other complex oscillations which are sensitive to initial conditions and are aperiodic in nature (*chaotic oscillations*). The molecular mechanisms and physico-chemical bases of rhythms form the link between chemical oscillations and biorhythms. A model biochemical pathway has been studied earlier in a collaborative project between C-MMACS and CCMB, Hyderabad. The model pathway is considered to be a part of a metabolic pathway in a cell and incorporates one positive feedback process coupled with one negative feedback process; an allosteric enzyme catalyses (positive feedback) the formation of the end product, S_3, which in turn controls its own production by inhibiting the first step in the pathway (negative feedback), the production of S_1, by an appropriate amount (see schematic presented as an inset in Fig. 4.3b). The mathematical model consists of a coupled system of non-linear ordinary differential equations in three variables.

Different domains in a two parameter space corresponding to steady states, limit cycles (through an Hopf bifurcation), period doublings, chaotic oscillations and reverse bifurcations to limit cycles are mapped for this system. The two parameter space are that of the degradation rates of S_1 and S_3.

In a continuation of the work, a simpler version of the model is derived and studied for its dynamical behaviours. Essentially, the term representing the allosteric reaction is further simplified keeping in mind the orders of magnitude of various factors appearing in the term. This has led to the realisation that the simplified system shares, by and large, the same range of dynamical behaviours and in approximately same areas of the parameter space.

In Fig. 4.3a the Hopf bifurcation surface of the original system is depicted, and in Fig. 4.3b that of the simplified system. It can be seen that the bifurcation surface is qualitatively similar. The surface in the parameter space across which period doublings take place is shown in Fig. 4.4 for the original system along with points marked in triangles where the corresponding behaviour occurs for the simplified system. *(T. R. Krishna Mohan).*

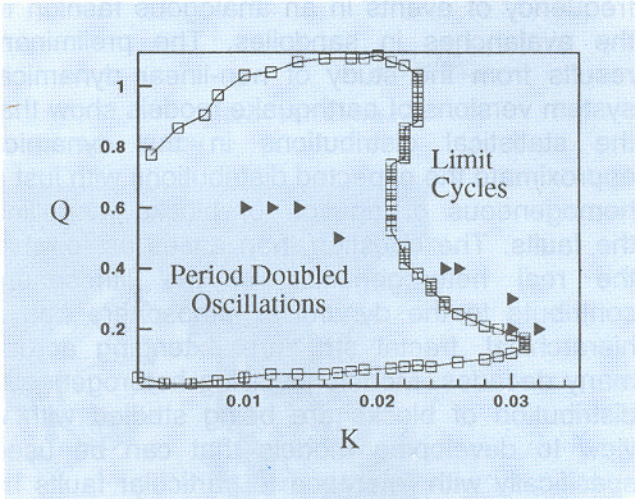

Fig. 4.4: Surface in the two parameter space across which period doublings take place in the original biochemical system. The triangles mark some of the points at which the same phenomenon takes place in the simplified version.